Effects of Boundary Conditions on the Free Vibrations of Circular Cylindrical Shells

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A simple formula for the natural frequency is derived as an asymptotic solution for the eigenvalue problems of the breathing type of free vibrations of a circular cylindrical shell. It is applicable under any possible combinations of the boundary conditions for the simply supported, the clamped, and the free ends. A characteristic value involved in the formula depends on the combination of three representative boundary conditions $SR(w = u = 0)$, $SF(w = N = 0)$, and $FR(N = S = 0)$, indicating that the free vibration characteristics depend on whether an end is free or supported and whether the supported end is allowed or not to move freely in the axial direction. The accuracy of the formula is examined by a comparison with numerical solutions and experimental results.

Introduction

It is widely accepted nowadays that the free vibration characteristics of circular cylindrical shells are affected by the boundary conditions, most significantly by those constraining the axial displacement. Arnold and Warburton[1,2] made a systematic attempt to clarify the effects of the boundary conditions by calculating approximate solutions under various boundary conditions of practical interest and by conducting an experiment. The importance of the boundary conditions imposed on the in-plane displacements and forces was first noted by Heki[3] in his memorable but unduely overlooked paper. The significance of the effect of the constraint on the axial displacement became more convincing after an extensive numerical analysis by Forsberg.[4,5] In principle, there exist 16 different sets of boundary conditions at an end. Forsberg calculated accurate numerical solutions for Flügge's equations in all possible cases of combinations of these sets and presented the results in 10 representative cases. One of the most important conclusions from his analysis on the breathing vibrations ($n \geq 2$) is that the effect of the constraint on the axial displacement is significant even for a long shell for all values of the thickness-to-radius ratio. As for the axisymmetric ($n = 0$) and the beam-like bending ($n = 1$) vibrations, the vibration characteristics are governed primarily by the membrane characteristics and strongly affected by the boundary conditions imposed on the circumferential as well as axial displacements and forces. A complete list of literature on the subject is given in a comprehensive survey by Leissa.[6]

In the present paper, a proof is given theoretically for the effect of the boundary conditions on the breathing vibrations. Asymptotic solutions are obtained for the eigenvalues assuming that the vibrations are nearly inextensional with the natural frequencies as low as those of the inextensional vibrations obtained by Rayleigh. Nine different sets of boundary conditions are considered, which include four variations for each of the simply supported and the clamped ends and one set for the free ends. The characteristic equations are derived for all 45 possible combinations of these nine sets of boundary conditions between the two ends. Only five different types of the characteristic equations result. An examination of the boundary conditions of each type reveals that there exist three sets of representative boundary conditions that determine the vibration characteristics. These representative boundary conditions clearly show that the vibration characteristics depend on whether an end is free or supported and whether the supported end is free or restrained in the axial direction. The asymptotic solutions also yield a simple formula for the natural frequency, whose accuracy is examined by a numerical comparison with accurate numerical solutions available and by an experiment conducted specifically for this purpose. The governing equations and the solution technique follow the developments in the author's previous paper[7] dealing with the inextensional vibrations, and the conclusion of this paper applies for all the breathing vibrations inclusive of the inextensional vibrations.

Governing Equations and Fundamental Assumptions

Let the radius, the thickness, and the length of a circular cylindrical shell be denoted by R, h, and $2L$, respectively. The shell is assumed to be made of an elastic material with Young's modulus E, Poisson's ratio ν, and the mass per unit volume ρ. The time is denoted by t. The axial coordinate x and the circumferential coordinate θ are set on the midsurface of the shell such that $-L \leqq x \leqq L$ and $0 \leqq \theta \leqq 2\pi$. The displacement components in the axial, circumferential, and lateral directions are denoted by u_x, u_θ, and w_z, respectively, w_z being positive for outward normal to the midsurface. The axial stress resultant and moment and the lateral and tangential components of the equivalent edge-shear are denoted by N_x, M_x, Q_x, and $S_{x\theta}$, respectively. Nondimensional quantities and operators are defined as follows:

$$
\begin{aligned}
y &= x/R \\
\ell &= L/R \\
T &= t/\mu \\
u &= u_x/R \\
v &= u_\theta/R \\
w &= w_z/R \\
N &= N_x/K \\
M &= RM_x/D \\
Q &= Q_x/K \\
S &= S_{x\theta}/K \\
(\ )' &= \partial(\ )/\partial y \\
(\ )^{\cdot} &= \partial(\ )/\partial\theta \\
(\ )^{*} &= \partial(\ )/\partial T \\
\nabla^2(\ ) &= (\ )'' + (\ )^{\cdot\cdot}
\end{aligned}
$$

where $K = Eh/(1-\nu^2)$, $D = Eh^3/12(1-\nu^2)$, and $\mu^2 = \rho h R^2/K$.

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Budiansky's equations[8] for small perturbations of stressed shells are specialized for the free vibrations of unstressed circular cylindrical shells. The equations of motion are written in terms of the displacements, which are then reduced to a single equation for w eliminating u and v. The result is

$$\nabla^8 w + 8w^{\prime\prime ::} + 2w^{:::} + (1-\nu^2)w^{\prime\prime\prime\prime}/\delta + 4w^{\prime\prime \cdot\cdot} + w^{::}$$
$$+ [\nabla^4 w - (3+2\nu)w^{\prime\prime} - w^{\cdot\cdot}]^{**}/\delta = 0 \qquad (1)$$

where δ is a small geometric parameter defined by

$$\delta = h^2/12R^2 \ll 1 \qquad (2)$$

Quantities to be prescribed as boundary conditions are expressed only in terms of w to yield the following relations:

$$\nabla^4 u = -\nu w^{\prime\prime\prime} + w^{\prime\cdot\cdot} \qquad (3a)$$

$$\nabla^4 v = -(2+\nu)w^{\prime\prime\cdot} - w^{\cdot\cdot\cdot} \qquad (3b)$$

$$\nabla^4 N = (1-\nu^2)w^{\prime\prime ::} + \delta\nu(w^{:::} + w^{::}) + \nu w^{\cdot\cdot **} \qquad (3c)$$

$$\nabla^4 M = -\nabla^4(w^{\prime\prime} + \nu w^{\cdot\cdot}) - \nu(2+\nu)w^{\prime\prime\cdot\cdot} - \nu w^{::} \qquad (3d)$$

$$\nabla^4 Q = -\nabla^4[w^{\prime\prime\prime} + (2-\nu)w^{\prime\cdot\cdot}] - 3w^{\prime\prime\prime\cdot\cdot} - (2-\nu)w^{\prime ::} \qquad (3e)$$

$$\nabla^4 S = -(1-\nu^2)w^{\prime\prime\prime\cdot} - \delta(2-\nu)w^{\prime :::}$$
$$-(2-\nu)\delta w^{\prime\cdot\cdot\cdot} + w^{\prime\cdot **} \qquad (3f)$$

The fundamental solution of Eq. (1) may be written in the form

$$w = \exp(\lambda y)\cos n\theta \sin\omega T \qquad (4)$$

where n is the circumferential wave number, ω is the frequency parameter, and λ is the eigenvalue that determines the modal characteristics along the generator. It should be noted here that Eqs. (1) and (3) have been derived under the assumptions

$$\delta|\lambda^2| \ll 1, \qquad \delta n^2 \ll 1, \qquad \omega^2 \ll 1 \qquad (5)$$

These are consistent with the fundamental assumptions of thin-shell theory and valid for highly flexural vibrations. In the previous paper, the author investigated the inextensional vibrations having the frequency ω_0:

$$\omega_0^2 = \delta n^2(n^2-1)^2/(n^2+1) \qquad (6)$$

which is known as Rayleigh's solution.

Let a geometric parameter Δ be defined and assumed much smaller than unity:

$$\Delta = n^2/2k^2 = \delta^{1/2} n^2/(1-\nu^2)^{1/2} \ll 1 \qquad (7)$$

It follows then that

$$\omega_0^2 = O(\Delta^2) \qquad (8)$$

In the present paper, we shall only be concerned with the free vibrations at such low frequencies that ω^2 is of the same order of magnitude as ω_0^2. It will be assumed, therefore, that

$$\omega^2 = O(\Delta^2) \qquad (9)$$

Eigenvalues

Substitution of Eq. (4) in Eq. (1) yields

$$\lambda^8 + A_3\lambda^6 + A_2\lambda^4 + A_1\lambda^2 + A_0 = 0 \qquad (10)$$

where

$$A_3 = -4n^2 \qquad (11a)$$

$$A_2 = (1-\nu^2)/\delta \qquad (11b)$$

$$A_1 = -4n^2(n^2-1)^2 + (2n^2+3+2\nu)\omega^2/\delta \qquad (11c)$$

$$A_0 = n^4(n^2-1)^2 - n^2(n^2+1)\omega^2/\delta \qquad (11d)$$

It can be shown that Eq. (10) has a positive and a negative root for λ^2, so that the roots for λ may be written as

$$\lambda_1, \lambda_2 = \pm n\xi_1 \qquad (12a)$$

$$\lambda_3, \lambda_4 = \pm in\eta_1 \qquad (12b)$$

$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm n(\xi_2 \pm i\eta_2) \qquad (12c)$$

where ξ_1, η_1, ξ_2, and η_2 are positive real and $i = (-1)^{1/2}$.

The root and coefficient relations of Eq. (10) read

$$2(\xi_2^2 - \eta_2^2) + (\xi_1^2 - \eta_1^2) = -A_3/n^2 \qquad (13a)$$

$$(\xi_2^2 + \eta_2^2)^2 - \xi_1^2\eta_1^2 + 2(\xi_2^2 - \eta_2^2)(\xi_1^2 - \eta_1^2) = A_2/n^4 \qquad (13b)$$

$$2\xi_1^2\eta_1^2(\xi_2^2 - \eta_2^2) - (\xi_1^2 - \eta_1^2)(\xi_2^2 + \eta_2^2)^2 = A_1/n^6 \qquad (13c)$$

$$\xi_1^2\eta_1^2(\xi_2^2 + \eta_2^2)^2 = -A_0/n^8 \qquad (13d)$$

Approximate solutions of Eqs. (13) are sought for by expanding ξ_1^2, η_1^2, ξ_2^2, and η_2^2 in asymptotic series in Δ such that

$$\xi_1^2 = \Delta(\xi_{10} + \Delta\xi_{11} + \Delta^2\xi_{12} + \dots) \qquad (14a)$$

$$\eta_1^2 = \Delta(\eta_{10} + \Delta\eta_{11} + \Delta^2\eta_{12} + \dots) \qquad (14b)$$

$$\xi_2^2 = \Delta^{-1}(\xi_{20} + \Delta\xi_{21} + \Delta^2\xi_{22} + \dots) \qquad (14c)$$

$$\eta_2^2 = \Delta^{-1}(\eta_{20} + \Delta\eta_{21} + \Delta^2\eta_{22} + \dots) \qquad (14d)$$

where ξ_{ij} and η_{ij} take real values of order-of-magnitude unity.

Substituting Eqs. (14) in Eqs. (13) and comparing terms of like powers in Δ on both sides, we obtain from Eqs. (13a–13c)

$$\xi_{10} = \eta_{10} \qquad (15a)$$

$$\xi_{20} = \eta_{20} = \tfrac{1}{2} \qquad (15b)$$

A first approximation is achieved by retaining only the leading terms of the series such that

$$\xi_1 = \eta_1 = \Delta^{1/2}\xi_0 + O(\Delta) \qquad (16a)$$

$$\xi_2 = \eta_2 = \Delta^{-1/2}/\sqrt{2} + O(\Delta) \qquad (16b)$$

where $\xi_0{}^2 = \xi_{10}$.

Consequently, the first approximation solutions for λ become

$$\lambda_1, \lambda_2, = \pm\Delta^{1/2} n\xi_0 \qquad (17a)$$

$$\lambda_3, \lambda_4, = \pm i\Delta^{1/2} n\xi_0 \qquad (17b)$$

$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm\Delta^{-1/2} n(1 \pm i)/\sqrt{2} \qquad (17c)$$

with errors of order-of-magnitude Δ. The eigenvalues λ_1, λ_2, λ_3, and λ_4 represent the global solutions that vary gradually over the entire surface of the shell, whereas λ_5, λ_6, λ_7, and λ_8 represent the edge-zone solutions that decay out rapidly as the distance from the end increases. In matrix notation, we write

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \Delta^{1/2} n \xi_0 \boldsymbol{g} \quad (18a)$$

$$(\lambda_5, \lambda_6, \lambda_7, \lambda_8) = \Delta^{-1/2} n \boldsymbol{e} \quad (18b)$$

where $\boldsymbol{g}$ and $\boldsymbol{e}$ are defined by

$$\boldsymbol{g}^m = [1, (-1)^m, i^m, (-i)^m] \quad (19a)$$

$$\boldsymbol{e}^m = \{(1+i)^m, (1-i)^m, [-(1+i)]^m, [-(1-i)]^m\}/2^{m/2} \quad (19b)$$

for $m = 1, 2, 3, \ldots$.

Having obtained the solutions of Eq. (10), we can write the general solution of Eq. (1) as

$$w = \left[\sum_{i=1}^{4} \exp(\lambda_i y) W_i + \sum_{j=5}^{8} \exp(\lambda_j y) W_j\right] \cos n\theta \sin \omega T \quad (20)$$

where W_m ($m = 1,2,3, \ldots, 8$) are arbitrary constants. In matrix notation, Eq. (20) is written as

$$w = (\mathbf{1}, \mathbf{1}) E(y) W \cos n\theta \sin \omega T \quad (21)$$

where

$$\mathbf{1} = (1, 1, 1, 1)$$

$$W = (W_g, W_e)$$

$$W_g = (W_1, W_2, W_3, W_4)^T$$

$$W_e = (W_5, W_6, W_7, W_8)^T$$

$$E(y) = \begin{pmatrix} E_g(y), & 0 \\ 0, & E_e(y) \end{pmatrix}$$

$$E_g(y) = \begin{bmatrix} \exp(\lambda_1 y), & 0, & 0, & 0 \\ & \exp(\lambda_2 y), & 0, & 0 \\ & & \exp(\lambda_3 y), & 0 \\ (\mathrm{sym}) & & & \exp(\lambda_4 y) \end{bmatrix}$$

$$E_e(y) = \begin{bmatrix} \exp(\lambda_5 y), & 0, & 0, & 0 \\ & \exp(\lambda_6 y), & 0, & 0 \\ & & \exp(\lambda_7 y), & 0 \\ (\mathrm{sym}) & & & \exp(\lambda_8 y) \end{bmatrix}$$

0 is a 4×4 null matrix and $(\ )^T$ means the transpose of ( ).

Also obtained from Eqs. (13) is a formula for the natural frequencies:

$$\omega^2 = \omega_0^2 [1 + (1 - \nu^2)\xi_1^4 / \delta(n^2 - 1)^2] \quad (22)$$

with errors of order-of-magnitude Δ. Equation (22) is identical with a formula derived by Nau and Simmonds[9] by asymptotic integration for shells with rigidly clamped ends ($C1$-$C1$ by our notation).

Boundary Conditions

Boundary conditions to be considered are those of the simply supported, the clamped, and the free ends. They are defined and designated as:

Simply supported ends

$$S1(w = M = u = v = 0) \quad (23a)$$

$$S2(w = M = u = S = 0) \quad (23b)$$

$$S3(w = M = N = v = 0) \quad (23c)$$

$$S4(w = M = N = S = 0) \quad (23d)$$

Clamped ends

$$C1(w = w' = u = v = 0) \quad (23e)$$

$$C2(w = w' = u = S = 0) \quad (23f)$$

$$C3(w = w' = N = v = 0) \quad (23g)$$

$$C4(w = w' = N = S = 0) \quad (23h)$$

Free ends

$$FR(Q = M = N = S = 0) \quad (23i)$$

Each of the homogeneous equations constituting the boundary conditions can be expressed in terms of W_m. The expressions for $w = 0$ and $w' = 0$ are readily obtained from Eq. (21). Those for the remaining equations are obtained from Eqs. (3): the right-hand members of Eqs. (3) are substituted from Eq. (21), and the particular solutions for those quantities on the left-hand side are sought for, neglecting small terms of order-of-magnitude Δ. The results are set equal to zero at $y = \ell$ and $-\ell$. Here, only those equations for $y = \ell$ are presented:

$$w = 0: \quad (\mathbf{1}, \mathbf{1}) E(\ell) W = 0 \quad (24a)$$

$$w' = 0: \quad (\Delta \xi_0 \boldsymbol{g}, \boldsymbol{e}) E(\ell) W = 0 \quad (24b)$$

$$u = 0: \quad (\xi_0 \boldsymbol{g}, -\nu \boldsymbol{e}^3) E(\ell) W = 0 \quad (24c)$$

$$v = 0: \quad [\Delta^{-1} \mathbf{1}, (2 + \nu) \boldsymbol{e}^3] E(\ell) W = 0 \quad (24d)$$

$$N = 0: \quad (\xi_0^2 \boldsymbol{g}^2, -\boldsymbol{e}^2) E(\ell) W = 0 \quad (24e)$$

$$M = 0: \quad [\Delta \nu (1 - 1/n^2) \mathbf{1}, -\boldsymbol{e}^2] E(\ell) W = 0 \quad (24f)$$

$$Q = 0: \quad [\Delta^2 \xi_0 (2 - \nu)(1 - 1/n^2) \boldsymbol{g}, -\boldsymbol{e}^3] E(\ell) W = 0 \quad (24g)$$

$$S = 0: \quad (\Delta \xi_0^3 \boldsymbol{g}^3, -\boldsymbol{e}^3) E(\ell) W = 0 \quad (24h)$$

The left-hand members of Eqs. (24) have been arranged in such a manner that the coefficients of the edge-zone solution part become of order-of-magnitude unity.

Systems of four simultaneous equations are formed from Eqs. (24) to constitute boundary conditions as specified in Eqs. (23). If the left-hand members of one equation are multiplied by an appropriate factor and subtracted by the corresponding members of another, either the global or edge-zone solution parts may cancel. The canceled parts do not vanish identically, but there remain residues of order-of-magnitude Δ. Thus, those residual parts may be written in matrix notation arbitrarily as

$$\Delta \boldsymbol{x} = \Delta(x_1, x_2, x_3, x_4) \quad (25)$$

where x_i are in general complex number whose absolute values are of order-of-magnitude unity, but whose specific values are indeterminate. Let us take for example a case where the boundary conditions at $y = \ell$ contain $w = 0$ and $v = 0$. Then, the system of four equations contains

$$w = 0: \ (\mathbf{1}, \mathbf{1}) E(\ell) W = 0$$

$$v = 0: \ [\Delta^{-1} \mathbf{1}, (2 + \nu) \boldsymbol{e}^2] E(\ell) W = 0$$

The left-hand members of the first equation are multiplied by Δ^{-1} and subsequently subtracted by those of the second equation. Then, the global solution part cancels out leaving a residue of $\Delta\boldsymbol{x}$ multiplied by Δ^{-1}, and the edge-zone solution part becomes $\Delta^{-1}\mathbf{1} - (2+\nu)\boldsymbol{e}^2$ in which $-(2+\nu)\boldsymbol{e}^2$ may be disregarded as a small term of order-of-magnitude Δ. The left-hand members of the resulting equation are multiplied by Δ to make the coefficient of the edge-zone solution part of order-of-magnitude unity. The equation for $w = 0$ now reads

$$w = 0\colon\ (\Delta\boldsymbol{x},\ \mathbf{1})\boldsymbol{E}(\ell)\boldsymbol{W} = 0$$

In similar manner, all of the systems of four simultaneous equations of the boundary conditions can be reduced to those involving $\boldsymbol{x}$. For the sake of space, only those parts of the coefficient matrices bracketed by ( ) are presented. The corresponding equations in Eqs. (24) are indicated by the quantities prescribed as boundary conditions. They are arranged in a decreasing order of the powers of Δ in the global solution parts.

$S1$	M:	$(\Delta^2\boldsymbol{x},\ \boldsymbol{e}^2)$	$S2$	M:	$(\Delta^2\boldsymbol{x},\ \boldsymbol{e}^2)$
	w:	$(\Delta\boldsymbol{x},\ \mathbf{1})$		S:	$(\Delta\xi_0^3\boldsymbol{g}^3,\ -\boldsymbol{e}^3)$
	u:	$(\xi_0\boldsymbol{g},\ -\nu\boldsymbol{e}^3)$		w:	$(\mathbf{1},\ \mathbf{1})$
	v:	$(\Delta^{-2}\mathbf{1},\ \boldsymbol{x})$		u:	$(\Delta^{-1}\boldsymbol{x},\ \mathbf{1})$
$S3$	M:	$(\Delta^2\boldsymbol{x},\ \boldsymbol{e}^2)$	$S4$	M:	$(\Delta^2\boldsymbol{x},\ \boldsymbol{e}^2)$
	w:	$(\Delta\boldsymbol{x},\ \mathbf{1})$		S:	$(\Delta\xi_0^3\boldsymbol{g}^3,\ \boldsymbol{e}^3)$
	N:	$(\Delta^{-1}\boldsymbol{g}^2,\ \boldsymbol{x})$		w:	$(\mathbf{1},\ \mathbf{1})$
	v:	$(\Delta^{-1}\mathbf{1},\ \nu\boldsymbol{e}^3)$		N:	$(\Delta^{-1}\boldsymbol{g}^2,\ \boldsymbol{x})$
$C1$	w':	$(\Delta^2\boldsymbol{x},\ \boldsymbol{e})$	$C2$	w':	$(\Delta^2\boldsymbol{x},\ \boldsymbol{e})$
	w:	$(\Delta\boldsymbol{x},\ \mathbf{1})$		S:	$(\Delta\xi_0^3\boldsymbol{g}^3,\ -\boldsymbol{e}^3)$
	u:	$(\xi_0\boldsymbol{g},\ -\nu\boldsymbol{e}^3)$		w:	$(\mathbf{1},\ \mathbf{1})$
	v:	$[\Delta^{-1}\mathbf{1},\ (2+\nu)\boldsymbol{e}^2]$		u:	$(\Delta^{-1}\boldsymbol{g},\ \boldsymbol{x})$
$C3$	w':	$(\Delta\xi_0\boldsymbol{g},\ \boldsymbol{e})$	$C4$	w':	$(\Delta\xi_0\boldsymbol{g},\ \boldsymbol{e})$
	w:	$(\Delta\boldsymbol{x},\ \mathbf{1})$		S:	$(\Delta\xi_0^3\boldsymbol{g}^3,\ -\boldsymbol{e}^3)$
	N:	$(\xi_0^2\boldsymbol{g}^2,\ -\boldsymbol{e}^2)$		w:	$(\mathbf{1},\ \mathbf{1})$
	v:	$(\Delta^{-2}\mathbf{1},\ \boldsymbol{x})$		N:	$(\xi_0^2\boldsymbol{g}^2,\ -\boldsymbol{e}^2)$

FR
$$Q\colon\ [\Delta^2(2-\nu)(1-1/n^2)\xi_0\boldsymbol{g},\ -\boldsymbol{e}^3]$$
$$M\colon\ [\Delta\nu(1-1/n^2)\mathbf{1},\ -\boldsymbol{e}^2]$$
$$S\colon\ (\xi_0^3\boldsymbol{g}^3,\ \boldsymbol{x})$$
$$N\colon\ (\Delta^{-1}\xi_0^2\boldsymbol{g}^2,\ \boldsymbol{x}) \qquad (26)$$

Characteristic Equations

There exist 45 possible combinations of the nine different sets of boundary conditions between the two ends. For each combination we have a system of eight simultaneous homogeneous equations for W_m; four from Eqs. (26) and four from the corresponding equations for $y = -\ell$. The characteristic equations are derived for all 45 combinations of the sets of boundary conditions by setting the determinants of the coefficient matrices equal to zero.

It is seen from Eqs. (26) that the global solution parts of the first two equations are of higher order of magnitude at least by Δ than those of the remaining two. Those higher-order parts are negligible within errors of order-of-magnitude Δ in the calculations of the characteristic determinants of the 8×8 coefficient matrices. The characteristic determinants, therefore, reduce in general to the form

$$\begin{vmatrix} \mathbf{0}, & \boldsymbol{B}_1 \\ \boldsymbol{G}, & \boldsymbol{B}_2 \end{vmatrix} = |\boldsymbol{G}|\,|\boldsymbol{B}_1|$$

where $\boldsymbol{G}$, $\boldsymbol{B}_1$, and $\boldsymbol{B}_2$ are 4×4 coefficient matrices; $\boldsymbol{G}$ consisting of the coefficients belonging to the global solution part and $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ of those belonging to the edge-zone solution part.

The determinants $|\boldsymbol{B}_1|$ are calculated from the edge-zone solution part for all 45 combinations of boundary conditions. The following four different values result:

$$|\boldsymbol{B}_1| = \pm 4\sqrt{2}(\sin 4k\ell \pm \sin 4k\ell),\ 8(\cos 4k\ell \pm \cosh 4k\ell),$$
$$-4(\cos 4k\ell + \cosh 4k\ell \pm 2) \qquad (27)$$

where k is a geometric parameter defined in Eq. (7).

Let us assume that the length of the shell is not much smaller than its diameter so that

$$1/n\ell = O(1) \qquad (28)$$

at most. Then, since

$$1/4k\ell = O(\Delta^{1/2}) \qquad (29)$$

none of the right-hand members of Eqs. (27) vanishes. It can now be concluded that $|\boldsymbol{B}_1| \neq 0$ and that the characteristic equations are given by $|\boldsymbol{G}| = 0$.

The characteristic determinants $|\boldsymbol{G}|$ can be calculated straightforward from the global solution part for all 45 combinations of boundary conditions. The following observations may be helpful, however, to have a better understanding of significant features of the effect of the in-plane boundary conditions.

It is seen from Eqs. (26) that the coefficient matrices in the global solution part for $w' = 0$, $M = 0$, and $Q = 0$ are of higher order in Δ, so that they have no contribution to $|\boldsymbol{G}|$. The coefficient matrices in the global solution part for $v = 0$, $u = 0$, and $N = 0$ are of lower order in Δ, indicating that they have a significant influence on the free vibration characteristics. In particular, those for $v = 0$ are of order of Δ^{-2}, lowest of all, and they have the most significant effect. It should be noted, however, that $v = 0$ are always accompanied by $w = 0$. Furthermore, the coefficient matrices in the global solution part are characterized by $\mathbf{1}$ in both $v = 0$ and $w = 0$, and they differ only by the factor Δ^{-1}. There is no difference, therefore, in the resulting characteristic equations no matter which of $v = 0$ and $w = 0$ is involved in $|\boldsymbol{G}|$. Consequently, we may regard $w = 0$ as a representative boundary condition for the supported ends without specifically referring to $v = 0$. Other predominant boundary conditions for the supported ends are $u = 0$ and $N = 0$, which are complementary to each other and whose coefficient matrices in the global solution part are characterized by $\boldsymbol{g}$ and $\boldsymbol{g}^2$, respectively. Consequently, it is sufficient for a supported end to specify either $w = u = 0$ or $w = N = 0$ in determining the free vibration characteristics. We shall refer to the supported end characterized by $w = u = 0$ as rigidly supported end to be designated by SR, and the one characterized by $w = N = 0$ as the freely supported end to be designated by SF.

For SR and SF the coefficient matrices contributing to $\boldsymbol{G}$ are

SR ($w = u = 0$):

$$\begin{bmatrix} \mathbf{1}\boldsymbol{E}_g(\ell) \\ \boldsymbol{g}\boldsymbol{E}_g(\ell) \end{bmatrix}$$

and SF ($w = N = 0$):

$$\begin{bmatrix} 1E_g(\ell) \\ g^2E_g(\ell) \end{bmatrix}$$

where E_g has been defined in definitions for Eq. (21).

As for the free ends designated by FR, the coefficient matrices in the global solution part for $Q = 0$ and $M = 0$ are of higher order in Δ, so that they are negligible in the calculations of the characteristic determinants. Only those for $N = 0$ and $S = 0$, characterized by g^2 and g^3, respectively, will contribute to G. Consequently, we have

FR ($N = S = 0$):

$$\begin{bmatrix} g^2E_g(\ell) \\ g^3E_g(\ell) \end{bmatrix}$$

The characteristic determinants $|G|$ are calculated for all possible combinations of SR, SF, and FR between $y = \ell$ and $-\ell$, and they are set equal to zero to derive the characteristic equations. It turns out that there are only four different types:

1) Type I: for SR-SR, (FR-FR)

$$\cosh 2n\xi_1\ell \cos 2n\xi_1\ell - 1 = 0 \tag{30}$$

2) Type II: for SR-SF, (SF-FR)

$$\cosh 2n\xi_1\ell \sin 2n\xi_1\ell - \sinh 2n\xi_1\ell \cos 2n\xi_1\ell = 0 \tag{31}$$

3) Type III: for SF-SF

$$\sin 2n\xi_1\ell = 0 \tag{32}$$

4) Type IV: for SR-FR

$$\cosh 2n\xi_1\ell \cos 2n\xi_1\ell + 1 = 0 \tag{33}$$

Equations (30-33) are identical in form with those characteristic equations for the free vibrations of a beam whose characteristic roots are known to be given as

$$2n\xi_1\ell \begin{cases} = 4.730,\ 7.853, \ldots & \text{(Type I)} \\ = 3.927,\ 7.069, \ldots & \text{(Type II)} \\ = \pi,\ 2\pi, \ldots & \text{(Type III)} \\ = 1.875,\ 4.694, \ldots & \text{(Type IV)} \end{cases} \tag{34}$$

Since $\xi_1 = O(\Delta^{1/2})$, Eqs. (34) imply that the shell must be long enough so as to be specified by

$$1/n\ell = O(\Delta^{1/2}) \tag{35}$$

It has been shown in the author's previous paper[7] that the inextensional vibrations occur under the boundary conditions designated here by FR-FR and SF-FR. These boundary conditions appearing in type I and II are enclosed in parentheses to indicate that they are applicable for the second and higher modes. The lowest modes under these boundary conditions are inextensional. The eigenvalues ξ_1 for the inextensional vibrations are characterized by $\xi_1^2 = 0$, which may be regarded as the characteristic equation of type V. Then, we have type V: for FR-FR, SF-FR

$$\xi_1^2 = 0 \tag{36}$$

The inextensional mode of vibrations can be represented by a deflection function which is constant or linear in y, if ℓ is bounded by

$$1/n\ell = O(1) \tag{37a}$$

and

$$\Delta n\ell = O(\Delta^{1/2}) \tag{37b}$$

When the free vibration characteristics of types I-V are discussed on a unified basis, we must employ Eq. (35) for the order-of-magnitude estimation of ℓ and anticipate errors of order-of-magnitude $\Delta^{1/2}$.

The characteristic equations and the combinations of the boundary conditions as well as those of the representative boundary conditions are summarized in Table 1.

Table 1 Characteristic equations and combinations of boundary conditions

Type	Characteristic equations	Combinations of boundary conditions	Representative boundary conditions
I	$\cosh 2n\xi_1\ell \cos 2n\xi_1\ell - 1 = 0$	$S1$-$S1$, $S1$-$S2$, $S1$-$C1$, $S1$-$C2$ $S2$-$S2$, $S2$-$C1$, $S2$-$C2$, $C1$-$C1$ $C1$-$C2$, $C2$-$C2$, (FR-FR)	SR-SR
II	$\cosh 2n\xi_1\ell \sin 2n\xi_1\ell - \sinh 2n\xi_1\ell \cos 2n\xi_1\ell = 0$	$S1$-$S3$, $S1$-$S4$, $S1$-$C3$, $S1$-$C4$ $S2$-$S3$, $S2$-$S4$, $S2$-$C3$, $S2$-$C4$ $C1$-$S3$, $C1$-$S4$, $C1$-$C3$, $C1$-$C4$ $C2$-$S3$, $C2$-$S4$, $C2$-$C3$, $C2$-$C4$ (FR-SF)	SR-SF
III	$\sin 2n\xi_1\ell = 0$	$S3$-$S3$, $S3$-$S4$, $S3$-$C3$, $S3$-$C4$ $S4$-$S4$, $S4$-$C3$, $S4$-$C4$, $C3$-$C3$ $C3$-$C4$, $C4$-$C4$	SF-SF
IV	$\cosh 2n\xi_1\ell \cos 2n\xi_1\ell + 1 = 0$	FR-$S1$, FR-$S2$, FR-$C1$, FR-$C2$	FR-SR
V	$\xi_1^2 = 0$	FR-FR FR-$S3$, FR-$S4$, FR-$C3$, FR-$C4$	FR-FR FR-SF

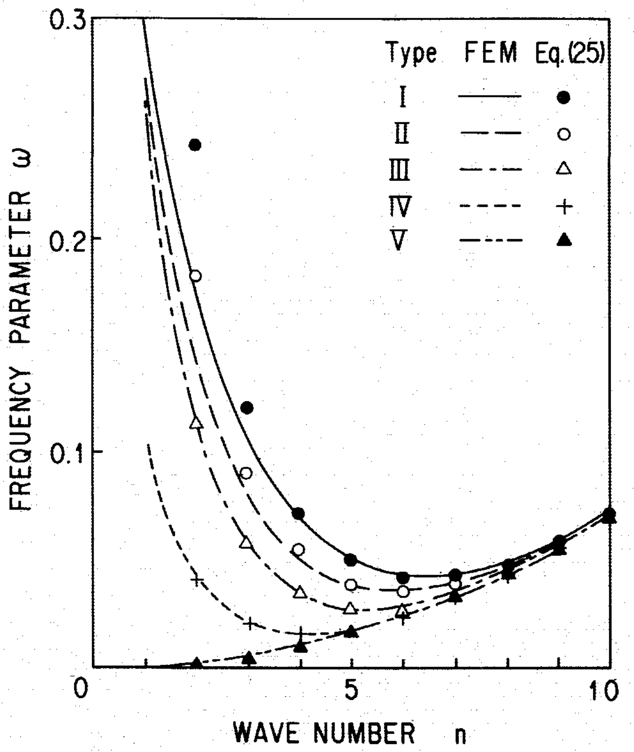

Fig. 1 Comparison with finite-element method solutions of first mode ($R/h = 400$, $2L/R = 4$, $\nu = 0.3$).

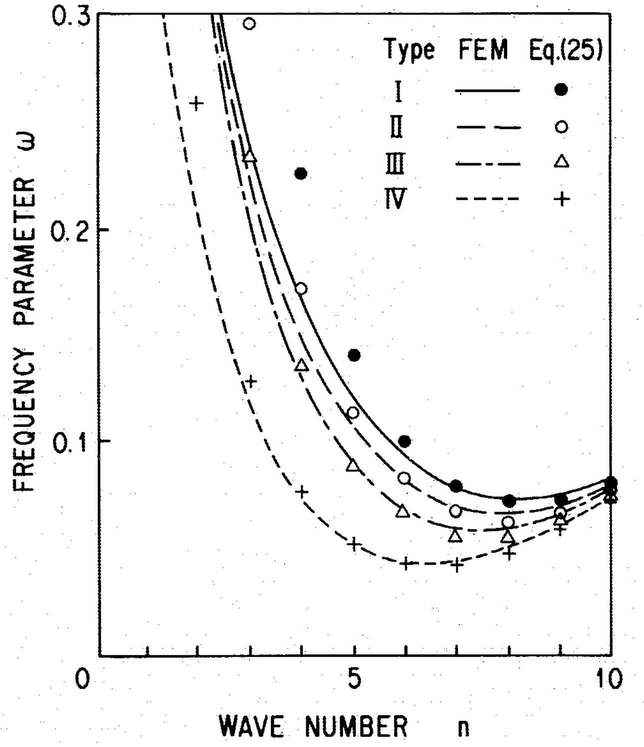

Fig. 2 Comparison with finite-element method solutions of second mode ($R/h = 400$, $2L/R = 4$, $\nu = 0.3$).

Comparison with Numerical Solutions and Experiment

The accuracy of the approximate solutions obtained in the preceding sections is examined by comparison with more accurate numerical solutions calculated by the finite-element method and with the results of an experiment that was conducted specifically for this purpose.

Numerical Solutions

A general purpose computer program based on the finite-element method for shells of revolution was used to calculate the natural frequencies of a circular cylindrical shell having $R/h = 400$, $2L/R = 4$, and $\nu = 0.3$. The computations were carried out for 10 different combinations of the boundary conditions; $C1$-$C1$, $C1$-$S1$, and $S1$-$S1$ for type I; $C1$-$S3$ and $S1$-$S3$ for type II; $S3$-$S3$ for type III; FR-$C1$ and FR-$S1$ for type IV; and FR-FR and FR-$S3$ for type V.

The results of the computations are shown by the curves in Figs. 1 and 2 together with the plots of the approximate solutions calculated from Eq. (22). The natural frequencies in Figs. 1 and 2 are for the first and second modes of vibrations having no and one circumferential nodal line, respectively. The approximate solutions for the first modes were calculated by using the lowest values of ξ_1 given in Eqs. (34), whereas those for the second modes were calculated by using the second from the lowest values. A good agreement is observed in general between the approximate solutions and the numerical solutions.

Experiment

An experiment was conducted on a seamless aluminum cylindrical shell having a radius of 33.0 mm and a thickness of 0.155 mm. The length varied in a range from 143 to 80 mm. It has a mass per unit volume of 3.23×10^3 kg/m^3 including the effect of dry powder spray paint that was painted over the outer surface of the cylinder to achieve a high optical reflection. Young's modulus of 69.0 GPa and Poisson's ratio of 0.3 were used for the calculations of the frequency parameter ω.

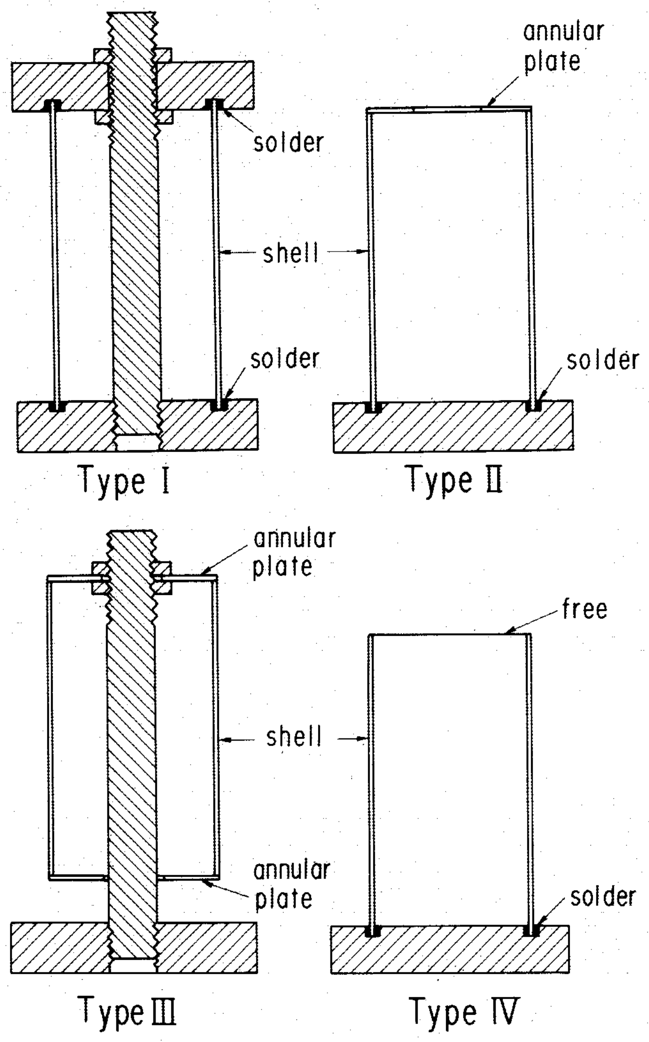

Fig. 3 Cross-sectional view of test specimens for types I–IV.

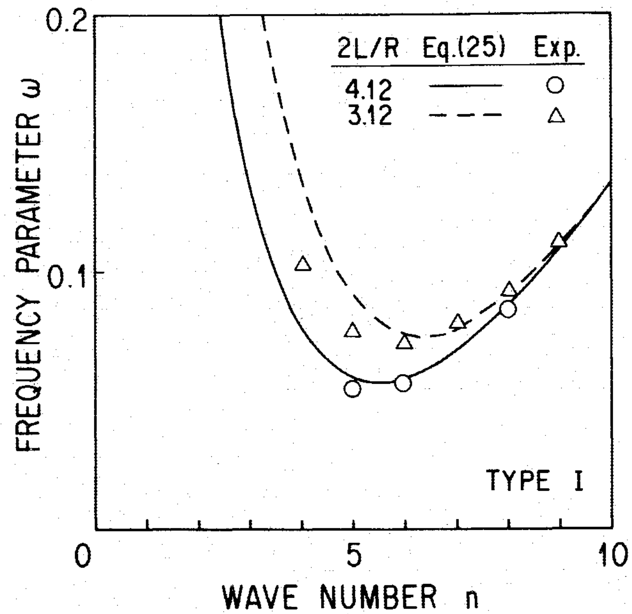

Fig. 4 Comparison with experiment, type I (R/h = 213).

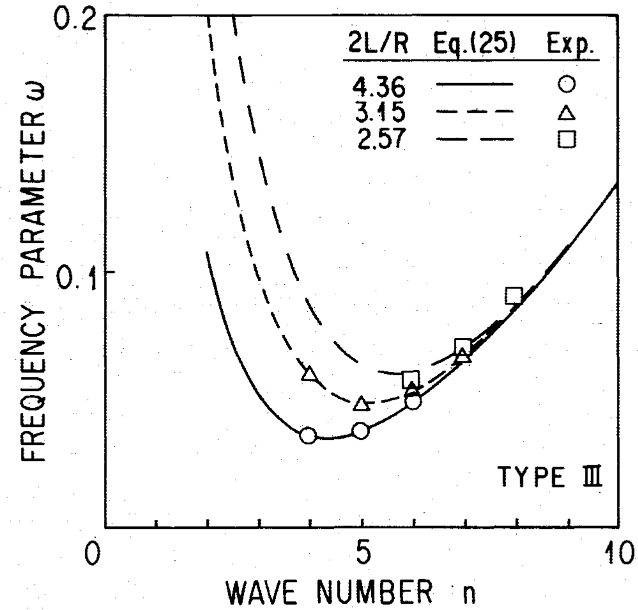

Fig. 6 Comparison with experiment, type III (R/h = 213).

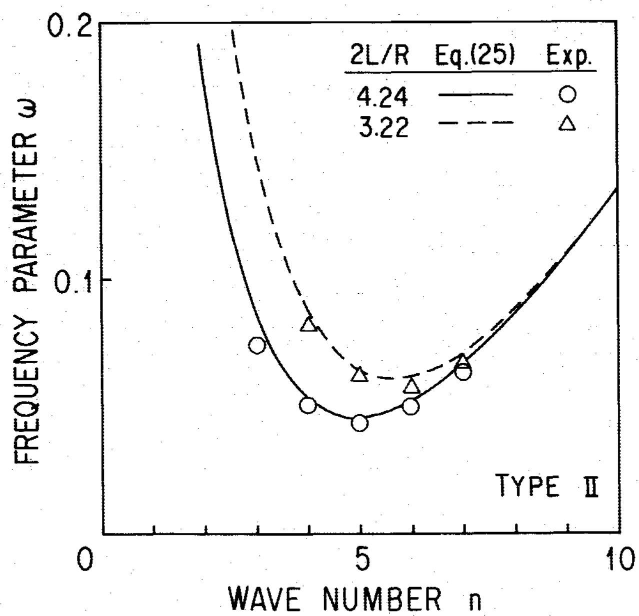

Fig. 5 Comparison with experiment, type II (R/h = 213).

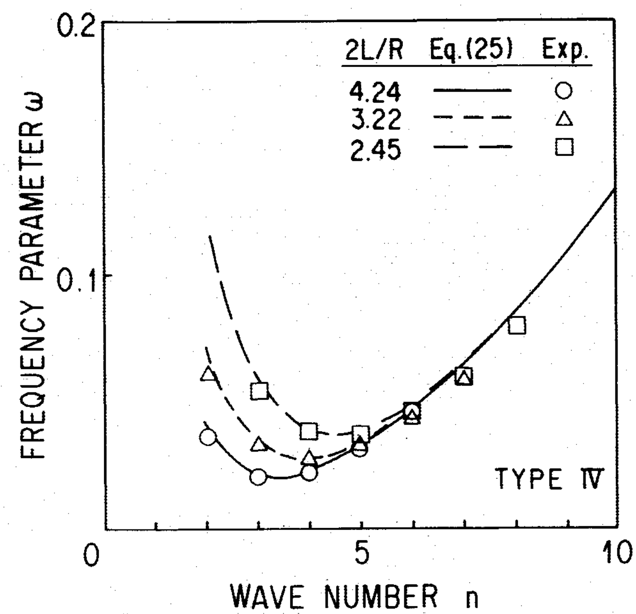

Fig. 7 Comparison with experiment, type IV (R/h = 213).

As shown in the preceding section, the free vibration characteristics depend on the representative boundary conditions: SR ($w = u = 0$), SF ($w = N = 0$), and FR ($N = S = 0$). To prove this by experiment we must establish in the test specimens those boundary conditions characterizing SR, SF, and FR. The boundary conditions $w = u = 0$ characterizing SR were established by soldering an end of the cylinder to a metal block by a low-melting-point metal. The boundary conditions $w = N = 0$ characterizing SF may be established by attaching a flexural thin annular plate at an end section. An annular plate having an inner radius of 11.0 mm and an outer radius of 33.0 mm was made from an aluminum sheet of a thickness of 0.30 mm, which was attached to an end by adhesive resin. The boundary conditions for FR were established naturally by leaving an end completely free. The cross-sectional view of the setups of the test specimens of types I–IV is shown in Fig. 3. Details of the experiment on type V including the setups of the test specimens were presented in Ref. 7.

The resonant vibrations were detected and their mode shapes visualized by holographic interferometry. The resonant frequencies of the first modes measured from the test specimens of types I–IV are plotted against n in Figs. 4–7, respectively. The curves in these figures are the theoretical values calculated with the aid of Eq. (22) taking n as a continuous variable. A satisfactory agreement is observed between theory and experiment. The same was observed for type V in Ref. 7.

Conclusions

The low natural frequencies of a circular cylindrical shell can be calculated with the aid of Eq. (22):

$$\omega^2 = \frac{\delta n^2(n^2-1)^2}{n^2+1}\left[1+\frac{(1-\nu^2)\xi_1^4}{\delta(n^2-1)^2}\right]$$

The formula is simple but accurate enough for engineering purpose. It is applicable under any possible combinations of the boundary conditions for the simply supported, the clamped, and the free ends, if use is made of the roots of the characteristic equations for ξ_1 according to the classification of Table 1. These characteristic values depend on the combinations of the representative boundary conditions: *SR* ($w = u = 0$), *SF* ($w = N = 0$), and *FR* ($N = S = 0$). It may be stated, therefore, that the free vibration characteristics of a circular cylindrical shell depend on whether an end is free or supported, and whether the supported end is allowed to move freely in the axial direction.

The present approximation for types I–IV is valid under the assumptions $\Delta \ll 1$ and $1/n\ell = O(\Delta^{1/2})$ with errors of order-of-magnitude Δ. If the inextensional vibrations are dealt with inclusively as belonging to type V, we must assume $\Delta^{1/2} \ll 1$ and anticipate errors of order-of-magnitude $\Delta^{1/2}$. A comparison with the accurate numerical solutions and experimental results obtained in the present paper as well as those available in various literature indicates that the present approximation is most likely to be valid in a range of $Z = (1-\nu^2)^{1/2}(2L/R)^2(R/h) \geq 200$.

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